

IMPNWU-9709011

Dynamically twisted algebra $A_{q,p;\hat{\pi}}(\widehat{gl}_2)$ as current algebra generalizing screening currents of q-deformed Virasoro algebra

Bo-yu Hou^b and Wen-li Yang^{a,b,*†}^a CCAST (World Laboratory), P.O.Box 8730 ,Beijing 100080, China^b Institute of Modern Physics, Northwest University, Xian 710069, China [‡]

February 9, 2008

Abstract

In this paper, we propose an elliptic algebra $A_{q,p;\hat{\pi}}(\widehat{gl}_2)$ which is based on the relations $RLL = LLR^*$, where R and R^* are the dynamical R-maxtrices of $A_1^{(1)}$ type face model with the elliptic moduli shifted by the center of the algebra. From the Ding-Frenkel correspondence, we find that its corresponding (Drinfeld) current algebra at level one is the algebra of screening currents for q-deformed Virasoro algebra. We realize the elliptic algebra at level one by Miki's construction from the bosonization for the type I and type II vertex operators. We also show that the algebra $A_{q,p;\hat{\pi}}(\widehat{gl}_2)$ is related with the algebra $A_{q,p}(\widehat{gl}_2)$ by a dynamically twisting.

1 Introduction

As the quantum form of fundamental Poisson bracket, the “ $RLL = LLR$ ” relations (or “ RLL ” formalism) define various quantum algebra which appear in quantum field theory (QFT) and statistical mechanics. It is associated with structure constants—R-matrix satisfying the Yang-Baxter equation. Drinfeld and Jimbo have discovered a fundamental algebra structure—quantum algebra $U_q(g)$ where g is some finite or infinite dimensional Lie algebra[3,4]. Faddeev, Reshetikhin and Takhtajan[5] realize the algebra $U_q(g)$

*e-mail :wlyang@nwu.edu.cn

†Fax :0086-029-8303511

‡Mailing address

(FRT construction), where \mathfrak{g} is some finite dimensional Lie algebra, by the “RLL” formalism with spectra parameter independent R-matrix. Later, Reshetikhin and Semenov-Tian-Shansky[6] constructed a new realization of q -deformed affine algebra by the “RLL” formalism with trigonometric R-matrix (which is so-called RS relations) characterized by the spectra parameter shifted with the center of the algebra. Ding and Frenkel[7] gave the isomorphism between the realization given by Reshetikhin and Semenov-Tian-Shansky and the Drinfeld realization of q -deformed affine algebra. Moreover, Khoroshkin[8] constructed successfully the realization of Yangian Double with center $DY_{\hbar}(\widehat{\mathfrak{g}})$ in the “RLL” formalism[9], which is associated with rational R-matrix. Foda et al proposed an elliptic extension of the quantum affine algebra $A_{q,p}(\widehat{sl_2})$ [10] as an symmetric algebra for the eight-vertex model. The elliptic algebra is based on the generalized “RLL” formalism : $RLL = LLR^*$, where R and R^* are eight-vertex R-matrices with elliptic moduli differing by an amount depending on the level k of the representation on which L acts. The algebra $A_{\hbar,\eta}(\widehat{sl_2})$ as the scaling limit of the elliptic algebra $A_{q,p}(\widehat{sl_2})$ was formulated in “ $RLL = LLR^*$ ” formulation by Jimbo et al[36] and was studied by Khoroshkin et al through the Gauss decomposition[8]. In fact, the above progress in “RLL” formalism which we mentioned are all involved in vertex type models.

Another progress of quantum algebra focus on the q -deformation [11-13] and \hbar -deformation[14] of Virasoro and W algebra, and q -deformed extended Virasoro algebra[27], which would play quite the same role in off-critical integrable model as that of Virasoro ,W algebra and extended Virasoro algebra[38-40] in two-dimensional conformal field theories (CFT)[15] for the critical model. Q -deformed Virasoro (q -Virasoro) algebra and q -deformed extended Virasoro(extended q -Virasoro) algebra arise in the two-dimensional solvable lattice models [12,17,27] (e.g ABF model[19] etc.); \hbar -deformed Virasoro (\hbar -Virasoro) algebra is studied as the hidden symmetry of the massive integrable field models (e.g. the Restricted sine-Gordon model [14]). It was shown that the screening currents for q -Virasoro algebra[12,13,20,21] and \hbar -Virasoro algebra [14] satisfy a closed algebra relations which is some further deformation of q -affine algebra and Yangian double with center. In another way , q -Virasoro algebra, extended q -Virasoro and \hbar -Virasoro algebra can be redefined as the algebra which commute with the screening currents up to a total difference[12-14,17,27]. Apparently, they constitute the hidden symmetries in $A_1^{(1)}$ type face model[12]. So, the studies of the algebraic structure of the screening currents for q -Virasoro, extended q -Virasoro and \hbar -Virasoro are of great importance. In this paper, we mainly deal with the screening algebra for q -Virasoro algebra and possibly the extended q -Virasoro ,as a byproduct, the screening algebra for \hbar -Virasoro algebra and related \hbar -deformed extended Virasoro algebra can be obtained by taking scaling limit of the q -deformed version.

The “RLL” formalism was originally formulated for the nondynamical Yang-Baxter equation (or the vertex type models), Felder[25] succeeded in extending it to incorporate dynamical Yang-Baxter equations (Gervais-Neveu-Felder equation)[22] which associate with q -deformation of Knizhnik-Zamolodchikov-Bernard equation on torus. In fact, the “RLL” formalism given by Felder[25] and Enriquez et al[22] is a dynamical version of the FRT construction and RS relations respectively. In this paper, we extend the works of Foda et al[10] to the dynamical R-matrix case. Namely, we proposed an elliptic algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ based on the relations $RLL = LLR^*$, where R and R^* are the dynamical R-matrices for $A_1^{(1)}$ face model (i.e the solution to Star-Triangle relation in $A_1^{(1)}$ type face model) with elliptic moduli shifted by the center of the algebra. Using the Ding-Frenkel correspondence, we construct Drinfeld currents for the algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$. From the Drinfeld currents for the algebra (which is a subalgebra of $A_{q,p;\hat{\pi}}(\widehat{gl_2})$), we show that the (Drinfeld) current algebra at level one and a higher level is just the algebra of screening currents for q -Virasoro algebra and the algebra of screening currents for the extended q -Virasoro algebra[27] respectively. The algebra of screening currents at level one was studied by Awata et al[12] for q -Virasoro algebra and by Feigin et al[13,21] for q -deformed W algebra, which is some elliptic deformation of affine algebra. The elliptic algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ at higher level would play an important role in the studies of the fusion ABF models[26,29] and relate with the extended q -Virasoro algebra. Moreover, the algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ is the dynamical twisted algebra[23-25] of the elliptic algebra $A_{q,p}(\widehat{gl_2})$.

The paper is organized as follows. In section 2, after some review of q -Virasoro algebra, we introduce the algebra of screening currents for q -Virasoro algebra. In section 3, we construct an elliptic algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ in terms of L^\pm -operator which satisfy the dynamical relations of $RLL = LLR^*$ formulation. From Ding-Frenkel correspondence, we find that the corresponding dynamical Drinfeld current form a subalgebra whose structure constants do not depend upon the dynamical variable and is just the algebra of screening currents for extended q -Virasoro algebra. The twisted relations between the algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ and $A_{q,p}(\widehat{gl_2})$ is constructed. In section 4, the bosonization of the type I and type II vertex operator for the algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ at level one is constructed. By the Miki’s construction, we obtained the bosonization for the algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ at level one. The corresponding generalizing algebra $A_{\hbar,\eta;\hat{\pi}}(\widehat{gl_2})$ for \hbar -deformed Virasoro algebra and \hbar -deformed extended Virasoro algebra, which is the scaling limit of the algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$, is studied in section 5. Finally, we give summary and discussions in section 6. Appendix contains some detailed calculations.

2 Algebra of screening currents for q-Virasoro algebra

We start with defining q-Virasoro algebra and corresponding quantum Miura transformation

2.1 q-Virasoro algebra and quantum Miura transformation

Let w be a generic complex number with $Im(w) > 0$ and r be a real number with $4 < r$, and set $x = e^{i\pi w}$.

Define elliptic functions

$$\begin{aligned}\theta \left[\begin{matrix} a \\ b \end{matrix} \right] (z, \tau) &= \sum_{m \in \mathbb{Z}} \exp\{i\pi[(m+a)^2\tau + 2(m+a)(z+b)]\} \quad , \quad Im(\tau) > 0 \\ \sigma_\alpha &= \sigma_{(\alpha_1, \alpha_2)}(z, \tau) = \theta \left[\begin{matrix} \frac{1}{2} + \frac{\alpha_1}{2} \\ \frac{1}{2} + \frac{\alpha_2}{2} \end{matrix} \right] (z, \tau) \\ \theta^{(k)}(z, \tau) &= \theta \left[\begin{matrix} -\frac{k}{2} \\ 0 \end{matrix} \right] (z, 2\tau)\end{aligned}$$

We will use the following abbreviation

$$\begin{aligned}[v]_t &= x^{\frac{v^2}{t} - v} \Theta_{x^{2t}}(x^{2v}) \quad , \quad = \sigma_0\left(\frac{v}{t}, -\frac{1}{tw}\right) \times const. \quad , \quad 0 < t \\ \Theta_q(z) &= (z; q)(qz^{-1}; q)(q; q) \quad , \quad (z; q_1, \dots, q_m) = \prod_{i_1, \dots, i_m=0}^{\infty} (1 - zq^{i_1} \dots q^{i_m})\end{aligned}$$

Q-Virasoro algebra generated by $\{T(z)\}$ with the following relations[11-13]

$$f\left(\frac{w}{z}\right)T(z)T(w) - f\left(\frac{z}{w}\right)T(w)T(z) = \frac{(x^r - x^{-r})(x^{(r-1)} - x^{-(r-1)})}{x - x^{-1}} \left(\delta\left(\frac{w}{x^2 z}\right) - \delta\left(\frac{x^2 w}{z}\right)\right) \quad (1)$$

where $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$ and

$$f(z) = (1 - z)^{-1} \frac{(zx^{2r}; x^4)(zx^{2-2r}; x^4)}{(zx^{2+2r}; x^4)(zx^{4-2r}; x^4)}$$

The generators $T(z)$ for q-Virasoro algebra can be obtained by the following quantum Miura transformation

$$T(z) = \Lambda(x^{-1}z) + \Lambda^{-1}(xz) \quad (2)$$

Define q-deformed bosonic oscillators β_m ($m \in \mathbb{Z}/\{0\}$)

$$[\beta_m, \beta_n] = m \frac{(x^m - x^{-m})(x^{(r-1)m} - x^{-(r-1)m})}{(x^{2m} - x^{-2m})(x^{rm} - x^{-rm})} \delta_{m+n, 0} \quad (3)$$

and zero mode operator P and Q such that $[P, iQ] = 1$

Then the fundamental operator $\Lambda(z)$ can be realized by q-deformed bosonic oscillators Eq.(3) as follows

$$\Lambda(z) = x\sqrt{2r(r-1)}^P : \exp\left\{-\sum_{m \neq 0} (x^{rm} - x^{-rm}) \frac{\beta_m}{m} z^{-m}\right\} : \quad (4)$$

2.2 Algebra of screening currents

Let us introduce the screening currents $E(v), F(v)$ for q-Virasoro algebra

$$E(v) = e^{i\sqrt{\frac{2(r-1)}{r}}(Q-i2v\ln x P)} : e^{\sum_{m \neq 0} \frac{x^m + x^{-m}}{m} \beta_m x^{-2vm}} : \quad (5)$$

$$F(v) = e^{-i\sqrt{\frac{2r}{r-1}}(Q-i2v\ln x P)} : e^{-\sum_{m \neq 0} \frac{x^m + x^{-m}}{m} \beta'_m x^{-2vm}} : \quad (6)$$

where $\beta'_m = \frac{x^{rm} - x^{-rm}}{x^{(r-1)m} - x^{-(r-1)m}} \beta_m$. Besides the well-known bosonic realization of screening currents $E(v), F(v)$, let us introduce H^\pm

$$H^-(v) = -x^{-4v} : E(v + \frac{1}{4}) F(v - \frac{1}{4}) : \quad (7)$$

$$H^+(v) = -x^{-4v} : E(v - \frac{1}{4}) F(v + \frac{1}{4}) : \quad (8)$$

The screening currents commute with the generators of q-Virasoro algebra up to a total difference and form a closed algebra. In fact, from the normal order in appendix A., one can find that the screening currents defined in Eq.(5)—Eq.(8) realize an algebra satisfying the relations

$$E(v_1)E(v_2) = \frac{[v_1 - v_2 - 1]_r}{[v_1 - v_2 + 1]_r} E(v_2)E(v_1) \quad (9)$$

$$F(v_1)F(v_2) = \frac{[v_1 - v_2 + 1]_{r-1}}{[v_1 - v_2 - 1]_{r-1}} F(v_2)F(v_1) \quad (10)$$

$$[E(v_1), F(v_2)] = \frac{1}{x - x^{-1}} \left\{ \delta(v_1 - v_2 + \frac{1}{2}) H^+(v_1 + \frac{1}{4}) - \delta(v_1 - v_2 - \frac{1}{2}) H^-(v_1 - \frac{1}{4}) \right\} \quad (11)$$

$$H^\pm(v_1)E(v_2) = \frac{[v_1 - v_2 - 1 \mp \frac{1}{4}]_r}{[v_1 - v_2 + 1 \mp \frac{1}{4}]_r} E(v_2)H^\pm(v_1) \quad (12)$$

$$H^\pm(v_1)F(v_2) = \frac{[v_1 - v_2 + 1 \pm \frac{1}{4}]_{r-1}}{[v_1 - v_2 - 1 \pm \frac{1}{4}]_{r-1}} F(v_2)H^\pm(v_1) \quad (13)$$

$$H^\pm(v_1)H^\pm(v_2) = \frac{[v_1 - v_2 + 1]_{r-1}[v_1 - v_2 - 1]_r}{[v_1 - v_2 - 1]_{r-1}[v_1 - v_2 + 1]_r} H^\pm(v_2)H^\pm(v_1) \quad (14)$$

$$H^+(v_1)H^-(v_2) = \frac{[v_1 - v_2 + 1 + \frac{1}{2}]_{r-1}[v_1 - v_2 - 1 - \frac{1}{2}]_r}{[v_1 - v_2 - 1 + \frac{1}{2}]_{r-1}[v_1 - v_2 + 1 - \frac{1}{2}]_r} H^-(v_2)H^+(v_1) \quad (15)$$

where $H^-(v) = H^+(v + \frac{1}{2} - r)$. The algebra of screening currents written by Awata[12] has the similar algebraic relations to ours. Actually, the screening currents defined in Eq.(9)—Eq.(15) realize the

(Drinfeld) current algebra of an elliptic algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ at level one, which will be given by “RLL” formalism in the following section .

3 The dynamical algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$

We propose an elliptic algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ based on a dynamical $RLL = LLR^*$ relations. Then by Ding-Frenkel correspondence, we will show that its Drinfeld current algebra is related to the current algebra generalizing the screening currents for q-Virasoro algebra. Moreover, algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ is an algebraic structure underlying the elliptic solution to the Star-Triangle relation in $A_1^{(1)}$ face type model including ABF[19] and its fused version[26,27,29].

3.1 The R-matrix

Define a dynamical elliptic R-matrix (R-matrix for $A_1^{(1)}$ face model[28,30])

$$R_F(v, \hat{\pi}) \equiv R_F(v, \hat{\pi}, r) = \begin{pmatrix} a & & & \\ & b & c & \\ & d & e & \\ & & & a \end{pmatrix} \quad (16)$$

where $\hat{\pi}$ is the dynamical variable corresponding the height for the face type model and enjoy in some relations with algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ (See Eq.(27) and Eq.(28)). The matrix elements of the R-matrix is defined by

$$a(v, \hat{\pi}) = x^{\frac{1-r}{r}v} \frac{g_1(v)}{g_1(-v)} \quad , \quad g_1(v) = \frac{\{x^{2+2v}\}\{x^{2+2r+2v}\}}{\{x^{4+2v}\}\{x^{2r+2v}\}} \quad , \quad z = (z; x^{2r}, x^4) \quad (17)$$

$$\frac{b(v, \hat{\pi})}{a(v, \hat{\pi})} = \frac{[v]_r[\hat{\pi} - 1]_r}{[v + 1]_r[\hat{\pi}]_r} \quad , \quad \frac{c(v, \hat{\pi})}{a(v, \hat{\pi})} = \frac{[v + \hat{\pi}]_r[1]_r}{[v + 1]_r[\hat{\pi}]_r} \quad (18)$$

$$\frac{d(v, \hat{\pi})}{a(v, \hat{\pi})} = \frac{[\hat{\pi} - v]_r[1]_r}{[v + 1]_r[\hat{\pi}]_r} \quad , \quad \frac{e(v, \hat{\pi})}{a(v, \hat{\pi})} = \frac{[v]_r[1 + \hat{\pi}]_r}{[v + 1]_r[\hat{\pi}]_r} \quad (19)$$

One can see that $a(v, \hat{\pi})$ does not depend on the dynamical variable $\hat{\pi}$. Moreover, let us introduce two R-matrices R_F^\pm which coincide with R_F in Eq.(16) up to scalar factors independent on the dynamical variable

$$R_F^\pm(v, \hat{\pi}) \equiv R_F^\pm(v, \hat{\pi}, r) = \tau^\pm(v) R_F(v, \hat{\pi}) \quad , \quad \tau^\pm(v) = \tau(-v \pm \frac{1}{2}) \quad (20)$$

$$\tau(v) = x^{-v} \frac{(x^{1+2v}; x^4)(x^{3-2v}; x^4)}{(x^{3+2v}; x^4)(x^{1-2v}; x^4)}$$

where $\tau^\pm(v)$ are the same as that of Foda et al [10]. R_F^\pm are regarded as linear operators on $V \otimes V$, with $V = \text{span}\{e_\pm\}$. Let h be the diagonal 2×2 matrix $\text{Diag}(1, -1)$ such that $he_\pm = \pm e_\pm$. The dynamical

R-matrices $R_F^\pm(v, \hat{\pi})$ satisfy the dynamical Yang-Baxter equation (i.e the modified Yang-Baxter equation [22]) in $V \otimes V \otimes V$

$$\begin{aligned} R_{F12}^\pm(v_1 - v_2, \hat{\pi} - 2h^{(3)}) R_{F13}^\pm(v_1 - v_3, \hat{\pi}) R_{F23}^\pm(v_2 - v_3, \hat{\pi} - 2h^{(1)}) \\ = R_{F23}^\pm(v_2 - v_3, \hat{\pi}) R_{F13}^\pm(v_1 - v_3, \hat{\pi} - 2h^{(2)}) R_{F12}^\pm(v_1 - v_2, \hat{\pi}) \end{aligned} \quad (21)$$

Here we choose the same notation as Enriquez et al[22]: $R_{F12}^\pm(v, \hat{\pi} - 2h^{(3)})$ means that if $a \otimes b \otimes e_\mu \in V \otimes V \otimes V, \mu \in \pm$, then $R_{F12}^\pm(v, \hat{\pi} - 2h^{(3)})a \otimes b \otimes e_\mu = R_{F12}^\pm(v, \hat{\pi} - 2\mu)a \otimes b \otimes e_\mu$, and the other symbols have a similar meaning. Besides the dynamical Yang-Baxter equation Eq.(21), the R-matrices have the following properties:

$$\textbf{Unitarity} \quad R_{F12}^\pm(v, \hat{\pi}) R_{F21}^\mp(-v, \hat{\pi}) = id \quad (22)$$

$$\textbf{Crossing relations} \quad R_F^\pm(-1 - v, \hat{\pi})_{\mu\nu}^{\mu'\nu'} = \mu\mu' R_F^\mp(v, \hat{\pi} - \mu')_{-\nu'\mu}^{-\nu\mu'} \frac{[\hat{\pi} - \mu']_r}{[\hat{\pi}]_r} \quad (23)$$

Moreover, the R-matrices $R_F^\pm(v, \hat{\pi})$ enjoys in the following property

$$R_F^+(v + r, \hat{\pi}) = R_F^-(v, \hat{\pi}) \quad (23a)$$

Here, if $R_{F12}^\pm(v, \hat{\pi}) = \sum a_i \otimes b_i$, with $a_i, b_i \in \text{End}(V)$, then $R_{F21}^\pm(v, \hat{\pi}) = \sum b_i \otimes a_i$.

3.2 The algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$

Let us proceed to the definition of the elliptic algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$. Consider L^\pm -operators

$$L^\pm(v, \hat{\pi}) = \begin{pmatrix} L_{++}^\pm & L_{+-}^\pm \\ L_{-+}^\pm & L_{--}^\pm \end{pmatrix}$$

which matrix elements are the generators of the elliptic algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ given by the following commutation relations:

$$R_F^+(v_1 - v_2 + \frac{c}{2}, \hat{\pi}) L_1^+(v_1, \hat{\pi}) L_2^-(v_2, \hat{\pi}) = L_2^-(v_2, \hat{\pi}) L_1^+(v_1, \hat{\pi}) R_F^{*+}(v_1 - v_2 - \frac{c}{2}, \hat{\pi}) \quad (24)$$

$$R_F^-(v_1 - v_2 - \frac{c}{2}, \hat{\pi}) L_1^-(v_1, \hat{\pi}) L_2^+(v_2, \hat{\pi}) = L_2^+(v_2, \hat{\pi}) L_1^-(v_1, \hat{\pi}) R_F^{*-}(v_1 - v_2 + \frac{c}{2}, \hat{\pi}) \quad (25)$$

$$R_F^\pm(v_1 - v_2, \hat{\pi}) L_1^\pm(v_1, \hat{\pi}) L_2^\pm(v_2, \hat{\pi}) = L_2^\pm(v_2, \hat{\pi}) L_1^\pm(v_1, \hat{\pi}) R_F^{*\pm}(v_1 - v_2, \hat{\pi}) \quad (26)$$

where $L_1^\pm(v, \hat{\pi}) = L^\pm(v, \hat{\pi}) \otimes id$, $L_2^\pm(v, \hat{\pi}) = id \otimes L^\pm(v, \hat{\pi})$, $R_F^{*\pm}(v, \hat{\pi}) = R_F^\pm(v, -\hat{\pi}, r - c)$ and c is the center of the algebra (its value on some representation of the algebra is usual called it as level of the algebra). Moreover, the L^\pm -operators are related with the dynamical variable $\hat{\pi}$:

$$\hat{\pi} L^\pm(v, \hat{\pi}) = L^\pm(v, \hat{\pi})(\hat{\pi} + (\nu r - (r - c)\mu))$$

Hence , the dynamical R-matrices have the following properties

$$R_F^\epsilon(v, \hat{\pi}) L^{(\epsilon')^\mu}_\nu(v, \hat{\pi}) = L^{(\epsilon')^\mu}_\nu(v, \hat{\pi}) R_F^\epsilon(v, \hat{\pi} + \mu c) \quad (27)$$

$$R_F^{*\epsilon}(v, \hat{\pi}) L^{(\epsilon')^\mu}_\nu(v, \hat{\pi}) = L^{(\epsilon')^\mu}_\nu(v, \hat{\pi}) R_F^{*\epsilon}(v, \hat{\pi} + \nu c) \quad (28)$$

where $\epsilon, \epsilon' \in \pm$ and the following property

$$[v + t]_t = -[v]$$

is used.

Remark: The relation Eq.(25) is the direct result of Eq.(22) and Eq.(24) .

Let

$$L^\pm(v, \hat{\pi}) = \begin{pmatrix} 1 & 0 \\ E^\pm(v) & 1 \end{pmatrix} \begin{pmatrix} K_1^\pm(v) & 0 \\ 0 & K_2^\pm(v) \end{pmatrix} \begin{pmatrix} 1 & F^\pm(v) \\ 0 & 1 \end{pmatrix}$$

be the Gauss decomposition of L^\pm -operators. For the convenience, we introduce the following symbols

$$R_F^\pm(v, \hat{\pi}) = \begin{pmatrix} a^\pm(v) & & & \\ & b^\pm(v) & c^\pm(v) & \\ & d^\pm(v) & e^\pm(v) & \\ & & & a^\pm(v) \end{pmatrix}, \quad R_F^{*\pm}(v, \hat{\pi}) = \begin{pmatrix} a'^\pm(v) & & & \\ & b'^\pm(v) & c'^\pm(v) & \\ & d'^\pm(v) & e'^\pm(v) & \\ & & & a'^\pm(v) \end{pmatrix}$$

Remark: The elements $a^\pm(v)$ and $a'^\pm(v)$ do not depend on the dynamical variable , and commute with the Gauss components of L^\pm -operators.

Define the total currents $E(v)$ and $F(v)$ by the corresponding Ding-Frenkel correspondence

$$E(v) = E^+(v) - E^-(v + \frac{c}{2}) \quad , \quad F(v) = F^+(v + \frac{c}{2}) - F^-(v) \quad (29)$$

Then we have the following

Proposition 1. The total currents $E(v), F(v)$ and $K_i^\pm(v)$ ($i = 1, 2$) satisfy the following commutation relations

$$a^\pm(v_1 - v_2) K_i^\pm(v_1) K_i^\pm(v_2) = K_i^\pm(v_2) K_i^\pm(v_1) a'^\pm(v_1 - v_2) \quad (30a)$$

$$b^\pm(v_1 - v_2) K_1^\pm(v_1) K_2^\pm(v_2) = K_2^\pm(v_2) K_1^\pm(v_1) b'^\pm(v_1 - v_2) \quad (30b)$$

$$a^+(v_1 - v_2 + \frac{c}{2}) K_i^+(v_1) K_i^-(v_2) = K_i^-(v_2) K_i^+(v_1) a'^\pm(v_1 - v_2 - \frac{c}{2}) \quad (30c)$$

$$b^+(v_1 - v_2 + \frac{c}{2}) K_1^+(v_1) K_2^-(v_2) = K_2^-(v_2) K_1^+(v_1) b'^+(v_1 - v_2 - \frac{c}{2}) \quad (30d)$$

$$b^-(v_1 - v_2 - \frac{c}{2}) K_1^-(v_1) K_2^+(v_2) = K_2^+(v_2) K_1^-(v_1) b'^-(v_1 - v_2 + \frac{c}{2}) \quad (30e)$$

$$K_1^+(v_1)E(v_2)K_1^+(v_1)^{-1} = \frac{a^+(v_1 - v_2)}{b^+(v_1 - v_2)}E(v_2) \quad (31a)$$

$$K_2^+(v_2)E(v_1)K_2^+(v_2)^{-1} = E(v_1)\frac{a^+(v_1 - v_2)}{b^+(v_1 - v_2)} \quad (31b)$$

$$K_1^-(v_1)E(v_2)K_1^-(v_1)^{-1} = \frac{a^+(v_1 - v_2 - \frac{c}{2})}{b^+(v_1 - v_2 - \frac{c}{2})}E(v_2) \quad (31c)$$

$$K_2^-(v_2)E(v_1)K_2^-(v_2)^{-1} = E(v_1)\frac{a^+(v_1 - v_2 + \frac{c}{2})}{b^+(v_1 - v_2 + \frac{c}{2})} \quad (31d)$$

$$K_1^+(v_1)^{-1}F(v_2)K_1^+(v_1) = F(v_2)\frac{a'^+(v_1 - v_2 - \frac{c}{2})}{b'^+(v_1 - v_2 - \frac{c}{2})} \quad (32a)$$

$$K_2^+(v_2)^{-1}F(v_1)K_2^+(v_2) = \frac{a'^+(v_1 - v_2 + \frac{c}{2})}{b'^+(v_1 - v_2 + \frac{c}{2})}F(v_1) \quad (32b)$$

$$K_1^-(v_1)^{-1}F(v_2)K_1^-(v_1) = F(v_2)\frac{a'^+(v_1 - v_2)}{b'^+(v_1 - v_2)} \quad (32c)$$

$$K_2^-(v_2)^{-1}F(v_1)K_2^-(v_2) = \frac{a'^+(v_1 - v_2)}{b'^+(v_1 - v_2)}F(v_1) \quad (32d)$$

$$E(v_1)\frac{a^\pm(v_1 - v_2)}{b^\pm(v_1 - v_2)}E(v_2) = E(v_2)\frac{a^\pm(v_2 - v_1)}{b^\pm(v_2 - v_1)}E(v_1) \quad (33a)$$

$$F(v_1)\frac{a'^\pm(v_2 - v_1)}{b'^\pm(v_2 - v_1)}F(v_2) = F(v_2)\frac{a'^\pm(v_1 - v_2)}{b'^\pm(v_1 - v_2)}F(v_1) \quad (33b)$$

$$\begin{aligned} [E(v_1), F(v_2)] &= \frac{1}{x - x^{-1}} \left\{ \delta(v_2 - v_1 - \frac{c}{2}) K_2^-(v_1 + \frac{c}{2}) \frac{[\hat{\pi}]_{r-c}[1]_{r-c}}{\theta'_{r-c}[\hat{\pi} - 1]_{r-c}} K_1^-(v_1 + \frac{c}{2})^{-1} \right. \\ &\quad \left. \delta(v_2 - v_1 + \frac{c}{2}) K_2^+(v_1) \frac{[\hat{\pi}]_{r-c}[1]_{r-c}}{\theta'_{r-c}[\hat{\pi} - 1]_{r-c}} K_1^+(v_1)^{-1} \right\} \end{aligned} \quad (34)$$

where $K_i^-(v) = K_i^+(v - \frac{c}{2} + r)$ and $\theta'_t = (x - x^{-1})\frac{\partial}{\partial v}[v]_t|_{v=0}$.

The proof of these relations is shifted to the appendix B.

Remark: The elliptic algebra $U_{q,p}(\widehat{sl_2})$ proposed by Konno[27], has different $K(v)$ from ours ($K_i^\pm(v)$ also in Ref.[37]), in which the commutation relations between $K(v)$ and $E(v), F(v)$ do not depend on the dynamical variable. However, they share the same subalgebra generated by $H^\pm(v), E(v)$ and $F(v)$ (see below) .

Set

$$H^+(v) = K_2^-(v + \frac{c}{4}) \frac{[\hat{\pi}]_{r-c}[1]_{r-c}}{\theta'_{r-c}[\hat{\pi} - 1]_{r-c}} K_1^-(v + \frac{c}{4})^{-1} \quad (35)$$

$$H^-(v) = K_2^+(v + \frac{c}{4}) \frac{[\hat{\pi}]_{r-c}[1]_{r-c}}{\theta'_{r-c}[\hat{\pi} - 1]_{r-c}} K_1^+(v + \frac{c}{4})^{-1} \quad (36)$$

Like the case of q-affine algebra[7] and Yangian double algebra[8,9], we can obtain the corresponding Drinfeld current algebra of $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ which is the subalgebra of the elliptic algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ and generated by $E(v), F(v), H^{\pm}(v)$. Namely, we have

Proposition 2. The elliptic (Drinfeld) current algebra of algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ is generated by $E(v), F(v), H^{\pm}(v)$ with the following algebraic relations

$$E(v_1)E(v_2) = \frac{[v_1 - v_2 - 1]_r}{[v_1 - v_2 + 1]_r} E(v_2)E(v_1) \quad (37)$$

$$F(v_1)F(v_2) = \frac{[v_1 - v_2 + 1]_{r-c}}{[v_1 - v_2 - 1]_{r-c}} F(v_2)F(v_1) \quad (38)$$

$$[E(v_1), F(v_2)] = \frac{1}{x - x^{-1}} \left\{ \delta(v_1 - v_2 + \frac{c}{2}) H^+(v_1 + \frac{c}{4}) - \delta(v_1 - v_2 - \frac{c}{2}) H^-(v_1 - \frac{c}{4}) \right\} \quad (39)$$

$$H^{\pm}(v_1)E(v_2) = \frac{[v_1 - v_2 - 1 \mp \frac{c}{4}]_r}{[v_1 - v_2 + 1 \mp \frac{c}{4}]_r} E(v_2)H^{\pm}(v_1) \quad (40)$$

$$H^{\pm}(v_1)F(v_2) = \frac{[v_1 - v_2 + 1 \pm \frac{c}{4}]_{r-c}}{[v_1 - v_2 - 1 \pm \frac{c}{4}]_{r-c}} F(v_2)H^{\pm}(v_1) \quad (41)$$

$$H^{\pm}(v_1)H^{\pm}(v_2) = \frac{[v_1 - v_2 + 1]_{r-c}[v_1 - v_2 - 1]_r}{[v_1 - v_2 - 1]_{r-c}[v_1 - v_2 + 1]_r} H^{\pm}(v_2)H^{\pm}(v_1) \quad (42)$$

$$H^+(v_1)H^-(v_2) = \frac{[v_1 - v_2 + 1 + \frac{c}{2}]_{r-c}[v_1 - v_2 - 1 - \frac{c}{2}]_r}{[v_1 - v_2 - 1 + \frac{c}{2}]_{r-c}[v_1 - v_2 + 1 - \frac{c}{2}]_r} H^-(v_2)H^+(v_1) \quad (43)$$

and

$$H^-(v) = H^+(v + \frac{c}{2} - r)$$

The proof of these formulas is shifted to the appendix C.

Remark: 1.The deformed parameters are $q = x$ and $p = x^{2r}$ (cf. Ref.[10]).Moreover, the constructure coefficient in Eq.(37)—Eq.(43) do not depend on the dynamical variable $\hat{\pi}$.

2.When $r \longrightarrow +\infty$, the limit current algebra is the algebra $U_q(\widehat{sl_2})$ [32].

One can see that if $c=1$ (i.e level one), the current algebra of algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ be the algebra of screening currents for q-Virasoro algebra (cf. Eq.(9)—Eq.(15)) which play the role of symmetry algebra in ABF model[12,17]. For the general level k ($k \in \text{integer}$),the algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ would correpond to the k -fusion ABF model[26,27,29] and in this case, some q-deformation of the extended Virasoro algebra[38-40] would exist in such a way that their screening currents satisfy current algebra of algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$. So, this elliptic algebra would play an important role in the studies of $A_1^{(1)}$ type face models as that of the algebra $A_{q,p}(\widehat{gl_2})$ in the eight-vertex model[10].

3.3 The algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ as the dynamically twisted algebra $A_{q,p}(\widehat{gl_2})$

It is well-known that there exists a face-vertex correspondence between $A_1^{(1)}$ face model and eight-vertex model when r is a generic one [26,28,29]. This would result in the “equivalence” between the underlying algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ and $A_{q,p}(\widehat{sl_2})$ — the algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ is the dynamcially twisted algebra of $A_{q,p}(\widehat{gl_2})$. In this section, we restrict our attention to the case of r being a generic one.

Let ϵ_μ ($\mu \in \pm$) be the orthonormal basis in R^2 , which are supplied with the inner product $\langle \epsilon_\mu, \epsilon_\nu \rangle = \delta_{\mu\nu}$. Set

$$\bar{\epsilon}_\mu = \epsilon_\mu - \epsilon_- \quad , \quad \epsilon = \frac{\epsilon_- + \epsilon_+}{2}$$

Then , define the intertwiners [28,33]

$$\begin{aligned} \varphi_{\hat{k},\mu}^{(m)}(v) &= \theta^{(m)}\left(\frac{v+ \langle \hat{k}, \bar{\epsilon}_\mu \rangle}{r}, -\frac{1}{rw}\right) \\ \varphi_{\mu,\hat{l}}^{\prime(m)}(v) &= \theta^{(m)}\left(\frac{v+ \langle \hat{l}, \bar{\epsilon}_\mu \rangle}{r-c}, -\frac{1}{(r-c)w}\right) \\ \langle \hat{k}, \bar{\epsilon}_\mu \rangle &\equiv \mu \hat{k} \quad , \quad \langle \hat{l}, \bar{\epsilon}_\mu \rangle \equiv \mu \hat{l} \\ \hat{\pi} &\equiv (r-c)\hat{k} - r\hat{l} \end{aligned} \tag{43a}$$

Here, we remark that the decomposition of Eq.(43a) can be defined only for the generic r [33] and

$$\hat{k}L^{(\pm)\mu}\nu(v, \hat{\pi}) = L^{(\pm)\mu}\nu(v, \hat{\pi})(\hat{k} + \mu c) \quad , \quad \hat{l}L^{(\pm)\mu}\nu(v, \hat{\pi}) = L^{(\pm)\mu}\nu(v, \hat{\pi})(\hat{l} + \nu c)$$

The face-vertex correspondence relations read as

$$\begin{aligned} R_{mn}^{ij}(v_1 - v_2) \varphi_{\hat{k},\nu}^{(m)}(v_1) \varphi_{\hat{k}-\bar{\epsilon}_\nu,\mu}^{(n)}(v_2) &= \sum_{\mu'\nu'} R_{F\nu'\mu'}^{\nu\mu}(v_1 - v_2, \hat{\pi}) \varphi_{\hat{k}-\bar{\epsilon}_{\mu'},\nu'}^{(i)}(v_1) \varphi_{\hat{k},\mu'}^{(j)}(v_2) \\ R_{mn}^{*ij}(v_1 - v_2) \varphi_{\nu,\hat{l}+\bar{\epsilon}_\mu}^{\prime(m)}(v_1) \varphi_{\mu,\hat{l}}^{\prime(n)}(v_2) &= \sum_{\mu\nu} R_{F\nu'\mu'}^{*\nu\mu}(v_1 - v_2, \hat{\pi}) \varphi_{\nu',\hat{l}}^{\prime(i)}(v_1) \varphi_{\mu',\hat{l}+\bar{\epsilon}_{\nu'}}^{\prime(j)}(v_2) \end{aligned}$$

where the nondynamical R -matrices R and R^* are the same as that of Foda et al [10]. Moreover, we can introduce intertwiners $\bar{\varphi}_{\hat{k},\mu}$ and $\bar{\varphi}'_{\mu,\hat{l}}$ satisfying relations[28]

$$\begin{aligned} \sum_m \bar{\varphi}_{\mu,\hat{k}}^{(m)} \varphi_{\nu,\hat{k}}^{(m)} &= \delta_{\mu,\nu} \quad , \quad \sum_\mu \bar{\varphi}_{\mu,\hat{k}}^{(i)} \varphi_{\mu,\hat{k}}^{(j)} = \delta^{ij} \\ \sum_m \bar{\varphi}'_{\hat{l},\mu}^{(m)} \varphi_{\hat{l},\nu}^{\prime(m)} &= \delta_{\mu,\nu} \quad , \quad \sum_\mu \bar{\varphi}'_{\hat{l},\mu}^{(i)} \varphi_{\hat{l},\mu}^{\prime(j)} = \delta^{ij} \end{aligned}$$

Then, we have the twisted relations between the R-matrix of eight-vertex model and the R-matrix of $A_1^{(1)}$ face model

$$R_{F\nu'\mu'}^{\nu\mu}(v_1 - v_2, \hat{\pi}) = \sum_{ijmn} \overline{\varphi}_{\hat{k},\mu'}^{(j)}(v_2) \overline{\varphi}_{\hat{k}-\bar{\epsilon}_{\mu'},\nu'}^{(i)}(v_1) R_{mn}^{ij}(v_1 - v_2) \varphi_{\hat{k},\nu}^{(m)}(v_1) \varphi_{\hat{k}-\bar{\epsilon}_{\nu},\mu}^{(n)}(v_2) \quad (44)$$

$$R_{F\nu'\mu'}^{*\nu\mu}(v_1 - v_2, \hat{\pi}) = \sum_{ijmn} \overline{\varphi}'_{\mu',\hat{l}+\bar{\epsilon}_{\nu'}}^{(j)}(v_2) \overline{\varphi}'_{\nu',\hat{l}}^{(i)}(v_1) R_{mn}^{*ij}(v_1 - v_2) \varphi'_{\nu,\hat{l}+\bar{\epsilon}_{\mu}}^{(m)}(v_1) \varphi'_{\mu,\hat{l}}^{(n)}(v_2) \quad (45)$$

Moreover, we can constructed the twisted relations between the corresponding L^{\pm} -operators

$$\begin{aligned} L_{\mu}^{\pm\nu}(v, \hat{\pi}) &= \sum_{mm'} \overline{\varphi}_{\hat{k},\nu}^{(m')} (v) L_m^{\pm m'}(v) \varphi_{\mu,\hat{l}}'^{(m)}(v) \\ L_m^{\pm m'}(v) &= \sum_{\mu\nu} \varphi_{\hat{k},\nu}^{(m')} (v) L_{\mu}^{\pm\nu}(v, \hat{\pi}) \overline{\varphi}_{\mu,\hat{l}}'^{(m)}(v) \end{aligned} \quad (46)$$

Then we have

Proposition 3. The $L^{\pm}(v)$ operators given by the twisted relations Eq.(46) satisfying the commutation relations of the algebra $A_{q,p}(\widehat{gl_2})$ [10]

$$R^{+}(v_1 - v_2 + \frac{c}{2}) L_1^{+}(v_1) L_2^{-}(v_2) = L_2^{-}(v_2) L_1^{+}(v_1) R^{*+}(v_1 - v_2 - \frac{c}{2}) \quad (47)$$

$$R^{\pm}(v_1 - v_2) L_1^{\pm}(v_1) L_2^{\pm}(v_2) = L_2^{\pm}(v_2) L_1^{\pm}(v_1) R^{*\pm}(v_1 - v_2) \quad (48)$$

where the r-matrices $R^{\pm}(v), R^{*\pm}(v)$ are the same as that of Foda et al

$$R^{\pm}(v) \equiv \tau^{\pm}(v) R(v) \quad , \quad R^{*\pm}(v) = R^{\pm}(v)|_{r \rightarrow r-c}$$

4 The type I and Type II vertex and Miki's construction

This section is devoted to the realization of an infinite dimensional representations of the algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ at level one by the q-primary fields of q-Virasoro algebra.

4.1 The type I and type II vertex operators

The method of bosonization provides a powerful method to study the solvable lattice model both in vertex type model[32] and the face type model[17,30,35]. In this subsection , we give the bosonization of the type I[17] and typ II[35] vertex operator in ABF model by one free field.

The type I vertex operator corresponds to the half-column transfer matrix of the model, and type II vertex operator is expected to creat the eigenstates of the transfer matrix. We denote the two types of vertex operator as

- Vertex operator of type I : $\Phi_i(v)$
- Vertex operator of type II : $\Psi_i^*(v)$

These vertex operators realize the Faddeev-Zamolodchikov (ZF) algebra with dynamical R-matrix as its structure coefficients

$$\Phi_\nu(v_2)\Phi_\mu(v_1) = R_{F\mu\nu}^{\mu'\nu'}(v_1 - v_2, \hat{\pi})\Phi_{\mu'}(v_1)\Phi_{\nu'}(v_2) \quad (49)$$

$$\Psi_\mu^*(v_1)\Psi_\nu^*(v_2) = -R_{F\mu'\nu'}^{*\mu\nu}(v_1 - v_2, \hat{\pi})\Psi_{\nu'}^*(v_2)\Psi_{\mu'}^*(v_1) \quad (50)$$

$$\Phi_\nu(v_1)\Psi_\mu^*(v_2) = \tau(v_1 - v_2)\Psi_\mu^*(v_2)\Phi_\nu(v_1) \quad (51)$$

Let us introduce the other basic operators

$$\begin{aligned} \eta_1(v) &= e^{-i\sqrt{\frac{r-1}{r}}(Q-2v\ln xP)} : e^{-\sum_{m \neq 0} \frac{\beta_m}{m} x^{-2vm}} : \\ \eta'_1(v) &= e^{i\sqrt{\frac{r}{r-1}}(Q-2v\ln xP)} : e^{\sum_{m \neq 0} \frac{\beta'_m}{m} x^{-2vm}} : \\ \xi(v) &= e^{i\sqrt{\frac{2(r-1)}{r}}(Q-2v\ln xP)} : e^{\sum_{m \neq 0} \frac{x^m + x^{-m}}{m} \beta_m x^{-2vm}} : \\ \xi'(v) &= e^{-i\sqrt{\frac{2r}{r-1}}(Q-2v\ln xP)} : e^{-\sum_{m \neq 0} \frac{x^m + x^{-m}}{m} \beta'_m x^{-2vm}} : \end{aligned}$$

where the q-deformed bosonic oscillators β_m, P, Q are defined in Eq.(3). Then, the bosonization of vertex operators are given by [17,30,33]

$$\Phi_+(v) = \eta_1(v) \quad , \quad \Phi_-(v) = \oint_C \frac{d(x^{2v_1})}{2\pi i x^{2v_1}} \eta_1(v) \xi(v_1) f(v_1 - v, \hat{\pi}) \quad (52)$$

$$\Psi_+(v) = \eta'_1(v) \quad , \quad \Psi_-(v) = \oint_{C'} \frac{d(x^{2v_1})}{2\pi i x^{2v_1}} \eta'_1(v) \xi'(v_1) f'(v_1 - v, \hat{\pi}) \quad (53)$$

where the integration contour C is a simple closed curves around the origin satisfying $|xx^{2v}| < |x^{2v_1}| < |x^{-1}x^{2v}|$; C' is chosen in such a way that the poles $x^{2v-1+2n(r-1)}$ ($0 \leq n$) are inside and the poles $x^{2v+1-2n(r-1)}$ ($0 \leq n$) are outside, and

$$f(v, w) = \frac{[v + \frac{1}{2} - w]_r}{[v - \frac{1}{2}]_r} \quad , \quad f'(v, w) = \frac{[v - \frac{1}{2} + w]_{r-1}}{[v + \frac{1}{2}]_{r-1}}$$

Set

$$\Psi_\mu^*(v) = \Psi_{-\mu}(v) \frac{1}{[\hat{\pi}]_{r-1}}$$

From the normal order relations given in appendix A, one can check that the bosonic realization for Φ_i and Ψ^* in Eq.(52)—Eq.(53) satisfy ZF algebra in Eq.(49)—Eq.(51) [30,33].

4.2 The realization of algebra $A_{q,p;\hat{\pi}}(\widehat{gl}_2)$ at level one by Miki's construction

Let us introduce Miki's construction[34]

$$L_{\nu}^{+\mu}(v, \hat{\pi}) = \Phi_{\mu}(v) \Psi_{\nu}^{*}(v - \frac{1}{2}) \quad (54)$$

$$L_{\nu}^{-\mu}(v, \hat{\pi}) = \Phi_{\mu}(v - \frac{1}{2}) \Psi_{\nu}^{*}(v) \quad (55)$$

Using the relations of ZF algebra in Eq.(49)—Eq.(51), one can prove the L^{\pm} -operators constructed above satisfy the definition of the elliptic quantum algebra Eq.(24)—Eq.(26) with $c = 1$. Moreover, we have

Proposition 4. The L^{\pm} -operators and two type vertex operators satisfy the following relations

$$R_F^{+}(v_1 - v_2, \hat{\pi}) L_1^{+}(v_1, \hat{\pi}) \Phi_2(v_2) = \Phi_2(v_2) L_1^{+}(v_1, \hat{\pi}) \quad (56)$$

$$R_F^{-}(v_1 - v_2 - \frac{1}{2}, \hat{\pi}) L_1^{-}(v_1, \hat{\pi}) \Phi_2(v_2) = \Phi_2(v_2) L_1^{-}(v_1, \hat{\pi}) \quad (57)$$

$$L_1^{+}(v_1, \hat{\pi}) \Psi_2^{*}(v_2) = \Psi_2^{*}(v_2) L_1^{+}(v_1, \hat{\pi}) R_F^{*+}(v_1 - v_2 - \frac{1}{2}, \hat{\pi}) \quad (58)$$

$$L_1^{-}(v_1, \hat{\pi}) \Psi_2^{*}(v_2) = \Psi_2^{*}(v_2) L_1^{-}(v_1, \hat{\pi}) R_F^{*+}(v_1 - v_2, \hat{\pi}) \quad (59)$$

The proof is direct by using Miki's construction of L^{\pm} -operators and ZF algebra Eq.(49)—Eq.(51).

From the Proposition 4, one can see that the vertex operators of ABF model are the intertwining operators of the elliptic algebra $A_{q,p;\hat{\pi}}(\widehat{gl}_2)$ at level one, which satisfy some generalized relations of q-affine algebra and its intertwining operators[31].

5 The scaling limit algebra $A_{\hbar,\eta;\hat{\pi}}(\widehat{gl}_2)$

Another deformed Virasoro algebra— \hbar -Virasoro algebra can be considered as symmetries of the massive integrable field theories[14], and at the semi-classical level corresponds to the center of the Yangian Double with center $DY_{\hbar}(\widehat{gl}_2)$ at the critical level[18]. In another way, \hbar -Virasoro algebra can be considered as the scaling limit of the q-Virasoro algebra[14]

$$x^{2v} = p^{-\frac{i\beta}{\hbar}} \quad , \quad q = p^{-\frac{1}{\eta}} \quad p \longrightarrow 1$$

Moreover, the screening currents of \hbar -Virasoro algebra satisfies a closed algebra relations, which also can be considered as the scaling limit of that of q-Virasoro algebra in Eq.(9)-Eq.(15)[14]. Therefore, we can construct a generalizing algebra $A_{\hbar,\eta;\hat{\pi}}(\widehat{gl}_2)$ as the scaling limit of algebra $A_{q,p;\hat{\pi}}(\widehat{gl}_2)$, which is expected to be the symmetric algebra of k-fused Restricted sine-Gordon model and its (Drinfeld) current algebra would be the algebra of screening currents of some \hbar -deformed extended Virasoro algebra. Similarly, the

algebra $A_{\hbar,\eta;\hat{\pi}}(\widehat{gl_2})$ can be formed in the dynamical $RLL = LLR^*$ form with the dynamical R-matrix being the trigonometric solution to the dynamical Yang-Baxter equation[14].

In this section, we restrict ourselves to the trigonometric dynamical R-matrix (or the scaling limit of R-matrix in Eq.(16) and Eq.(20)). Without confusion with that in former section, we choose the same symbols as that of section 3.1

$$R_F(v, \hat{\pi}) \equiv R_F(v, \hat{\pi}, \eta) = \begin{pmatrix} a & & & \\ & b & c & \\ & d & e & \\ & & & a \end{pmatrix} \quad (60)$$

and

$$a(\beta, \hat{\pi}) = \kappa(\beta) = \exp\left\{ \int_0^\infty \frac{2sh\frac{\hbar t}{2} sh\frac{\hbar t}{2\eta} shi\beta t}{sh\hbar t sh\frac{(1+\eta)\hbar t}{2\eta}} \frac{dt}{t} \right\} \quad (61)$$

$$\frac{b(\beta, \hat{\pi})}{a(\beta, \hat{\pi})} = \frac{\sin\pi\eta(\frac{i\beta}{\hbar})\sin\pi\eta(\hat{\pi}-1)}{\sin\pi\eta(\hat{\pi})\sin\pi\eta(\frac{i\beta}{\hbar}+1)}, \quad \frac{c(\beta, \hat{\pi})}{a(\beta, \hat{\pi})} = \frac{\sin\pi\eta\sin\pi\eta(\frac{i\beta}{\hbar}+\hat{\pi})}{\sin\pi\eta(\hat{\pi})\sin\pi\eta(\frac{i\beta}{\hbar}+1)} \quad (62)$$

$$\frac{d(\beta, \hat{\pi})}{a(\beta, \hat{\pi})} = \frac{\sin\pi\eta\sin\pi\eta(-\frac{i\beta}{\hbar}+\hat{\pi})}{\sin\pi\eta(\hat{\pi})\sin\pi\eta(\frac{i\beta}{\hbar}+1)}, \quad \frac{e(\beta, \hat{\pi})}{a(\beta, \hat{\pi})} = \frac{\sin\pi\eta(\frac{i\beta}{\hbar})\sin\pi\eta(\hat{\pi}+1)}{\sin\pi\eta(\hat{\pi})\sin\pi\eta(\frac{i\beta}{\hbar}+1)} \quad (63)$$

$$R_F^\pm(\beta, \hat{\pi}) = \tau^\pm(\beta) R_F(\beta, \hat{\pi}), \quad \tau^+(\beta) = ctg(\frac{i\pi\beta}{2\hbar}), \quad \tau^-(\beta) = -tg(\frac{i\pi\beta}{2\hbar})$$

Define

$$R_F^{*\pm}(\beta, \hat{\pi}) \equiv R_F^\pm(\beta, -\hat{\pi})|_{\eta \rightarrow \eta'}, \quad \frac{1}{\eta'} = \frac{1}{\eta} - c$$

The algebra $A_{\hbar,\eta;\hat{\pi}}(\widehat{gl_2})$ is generated by the matrices elements of L^\pm -operator which satisfy the following relations

$$R_F^+(\beta_1 - \beta_2 - \frac{i\hbar c}{2}, \hat{\pi}) L_1^+(\beta_1, \hat{\pi}) L_2^-(\beta_2, \hat{\pi}) = L_2^-(\beta_2, \hat{\pi}) L_1^+(\beta_1, \hat{\pi}) R_F^{*+}(\beta_1 - \beta_2 + \frac{i\hbar c}{2}, \hat{\pi}) \quad (64)$$

$$R_F^-(\beta_1 - \beta_2 + \frac{i\hbar c}{2}, \hat{\pi}) L_1^-(\beta_1, \hat{\pi}) L_2^+(\beta_2, \hat{\pi}) = L_2^+(\beta_2, \hat{\pi}) L_1^-(\beta_1, \hat{\pi}) R_F^{*-}(\beta_1 - \beta_2 - \frac{i\hbar c}{2}, \hat{\pi}) \quad (65)$$

$$R_F^\pm(\beta_1 - \beta_2, \hat{\pi}) L_1^\pm(\beta_1, \hat{\pi}) L_2^\pm(\beta_2, \hat{\pi}) = L_2^\pm(\beta_2, \hat{\pi}) L_1^\pm(\beta_1, \hat{\pi}) R_F^{*\pm}(\beta_1 - \beta_2, \hat{\pi}) \quad (66)$$

$$\hat{\pi} L^{(\pm)\mu}_\nu(\beta, \hat{\pi}) = L^{(\pm)\mu}_\nu(\beta, \hat{\pi}) (\hat{\pi} + (\nu \frac{1}{\eta} - \frac{1}{\eta'} \mu)) \quad (67)$$

Let

$$L^\pm(\beta, \hat{\pi}) = \begin{pmatrix} 1 & 0 \\ E^\pm(\beta) & 1 \end{pmatrix} \begin{pmatrix} K_1^\pm(\beta) & 0 \\ K_2^\pm(\beta) & 1 \end{pmatrix} \begin{pmatrix} 1 & F^\pm(\beta) \\ 0 & 1 \end{pmatrix}$$

be the Gauss decomposition of L^\pm -operators. For the convenience, we also introduce the following symbols

$$R_F^\pm(\beta, \hat{\pi}) = \begin{pmatrix} a^\pm(\beta) & & & \\ & b^\pm(\beta) & c^\pm(\beta) & \\ & d^\pm(\beta) & e^\pm(\beta) & \\ & & & a^\pm(\beta) \end{pmatrix}, \quad R_F^{*\pm}(\beta, \hat{\pi}) = \begin{pmatrix} a'^\pm(\beta) & & & \\ & b'^\pm(\beta) & c'^\pm(\beta) & \\ & d'^\pm(\beta) & e'^\pm(\beta) & \\ & & & a'^\pm(\beta) \end{pmatrix}$$

Define the total currents $E(\beta)$ and $F(\beta)$ by the corresponding Ding-Frenkel correspondence

$$E(\beta) = E^+(\beta) - E^-(\beta - \frac{i\hbar c}{2}) \quad , \quad F(\beta) = F^+(\beta - \frac{i\hbar c}{2}) - F^-(\beta) \quad (68)$$

Substituting the Gauss decomposition of L^\pm -operator, we can obtain the similar commutation relations between $E(\beta), F(\beta)$ and $K_i^\pm(\beta)$ as those in the proposition 1, where the matrices elements of R-matrix are in Eq.(60)—Eq.(63).

The algebra $A_{\hbar, \eta}(\widehat{gl_2})$ as the scaling limit of the elliptic algebra $A_{q, p}(\widehat{gl_2})$ was studied by Khoroshkin et al through the method of Gauss decomposition[8]. The commutation relations of $E(\beta), F(\beta)$ and $K_i^\pm(\beta)$ were obtained, which are quite different from ours. This is due to that they chose a nondynamical “ RLR^* ” formalism. Consequently, the commutation relations of theirs do not depend on the dynamical variable. If we introduce $H^\pm(\beta)$ as Eq.(69) and Eq.(70), which quite differ from that of Khoroshkin et al, algebra $A_{\hbar, \eta; \hat{\pi}}(\widehat{gl_2})$ and algebra $A_{\hbar, \eta}(\widehat{gl_2})$ share the same subalgebra—the (Drinfeld) current algebra of each one generated by $E(\beta), F(\beta), H^\pm(\beta)$. Although they share the same subalgebraic commutation relations, they yet relate to the different E, F, K_i^\pm (or the different algebra) and consequently are associated with different vertex operators[18,14]. Moreover, the two algebra are related with the different models (face type model for the dynamical algebra and vertex model for the nondynamical algebra) especially for the rational η case.

Set

$$H^+(\beta) = K_2^-(\beta - \frac{i\hbar c}{4}) \frac{\sin \pi \eta' \hat{\pi} \sin \pi \eta'}{\pi \eta' \sin \pi \eta' (\hat{\pi} - 1)} K_1^-(\beta - \frac{i\hbar c}{4})^{-1} \quad (69)$$

$$H^-(\beta) = K_2^+(\beta - \frac{i\hbar c}{4}) \frac{\sin \pi \eta' \hat{\pi} \sin \pi \eta'}{\pi \eta' \sin \pi \eta' (\hat{\pi} - 1)} K_1^+(\beta - \frac{i\hbar c}{4})^{-1} \quad (70)$$

we have the current algebra of algebra $A_{\hbar, \eta; \hat{\pi}}(\widehat{gl_2})$ generated by $E(\beta), F(\beta)$ and $H^\pm(\beta)$ which is the scaling limit of those of the elliptic algebra $A_{q, p; \hat{\pi}}(\widehat{gl_2})$.

Proposition 5. The current algebra of algebra $A_{\hbar, \eta; \hat{\pi}}(\widehat{gl_2})$ is generated by $E(\beta), F(\beta), H^\pm(\beta)$ and satisfies the following relations

$$E(\beta_1)E(\beta_2) = \frac{\sin \frac{i\pi\eta}{\hbar}(\beta_1 - \beta_2 - i\hbar)}{\sin \frac{i\pi\eta}{\hbar}(\beta_1 - \beta_2 + i\hbar)} E(\beta_2)E(\beta_1) \quad (71)$$

$$F(\beta_1)F(\beta_2) = \frac{\sin \frac{i\pi\eta'}{\hbar}(\beta_1 - \beta_2 + i\hbar)}{\sin \frac{i\pi\eta'}{\hbar}(\beta_1 - \beta_2 - i\hbar)} F(\beta_2)F(\beta_1) \quad (72)$$

$$[E(\beta_1), F(\beta_2)] = \hbar \left\{ \delta(\beta_1 - \beta_2 - \frac{i\hbar c}{2}) H^+(\beta_1 - \frac{i\hbar c}{4}) - \delta(\beta_1 - \beta_2 + \frac{i\hbar c}{2}) H^-(\beta_1 + \frac{i\hbar c}{4}) \right\} \quad (73)$$

$$H^\pm(\beta_1)E(\beta_2) = \frac{\sin \frac{i\pi\eta}{\hbar}(\beta_1 - \beta_2 - i\hbar \mp \frac{i\hbar c}{4})}{\sin \frac{i\pi\eta}{\hbar}(\beta_1 - \beta_2 + i\hbar \mp \frac{i\hbar c}{4})} E(\beta_2)H^\pm(\beta_1) \quad (74)$$

$$H^\pm(\beta_1)F(\beta_2) = \frac{\sin \frac{i\pi\eta'}{\hbar}(\beta_1 - \beta_2 + i\hbar \pm \frac{i\hbar c}{4})}{\sin \frac{i\pi\eta'}{\hbar}(\beta_1 - \beta_2 - i\hbar \pm \frac{i\hbar c}{4})} F(\beta_2)H^\pm(\beta_1) \quad (75)$$

$$H^\pm(\beta_1)H^\pm(\beta_2) = \frac{\sin \frac{i\pi\eta'}{\hbar}(\beta_1 - \beta_2 + i\hbar) \sin \frac{i\pi\eta}{\hbar}(\beta_1 - \beta_2 - i\hbar)}{\sin \frac{i\pi\eta'}{\hbar}(\beta_1 - \beta_2 - i\hbar) \sin \frac{i\pi\eta}{\hbar}(\beta_1 - \beta_2 + i\hbar)} H^\pm(\beta_2)H^\pm(\beta_1) \quad (76)$$

$$H^+(\beta_1)H^-(\beta_2) = \frac{\sin \frac{i\pi\eta'}{\hbar}(\beta_1 - \beta_2 + i\hbar - \frac{i\hbar c}{2}) \sin \frac{i\pi\eta}{\hbar}(\beta_1 - \beta_2 - i\hbar + \frac{i\hbar c}{2})}{\sin \frac{i\pi\eta'}{\hbar}(\beta_1 - \beta_2 - i\hbar - \frac{i\hbar c}{2}) \sin \frac{i\pi\eta}{\hbar}(\beta_1 - \beta_2 + i\hbar + \frac{i\hbar c}{2})} H^-(\beta_2)H^+(\beta_1) \quad (77)$$

$$(78)$$

It can be seen that when $c = 1$ (i.e at level one), the current algebra of algebra $A_{\hbar,\eta;\hat{\pi}}(\widehat{gl_2})$ be the algebra of the screening currents for \hbar -Virasoro algebra[14]. For higer level,it would be the algebra of screening currents for \hbar -deformed extented Virasoro algebra. Moreover, there exist the following relations between the algebra $A_{\hbar,\eta;\hat{\pi}}(\widehat{gl_2})$ and $A_{\hbar,\eta}(\widehat{gl_2})$ [8],and between the algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ and $A_{q,p}(\widehat{gl_2})$ [10] for the generic r and η case

$$\begin{array}{ccc} A_{q,p;\hat{\pi}}(\widehat{gl_2}) & \xrightarrow{\text{scaling limit}} & A_{\hbar,\eta;\hat{\pi}}(\widehat{gl_2}) \\ \uparrow \text{twisted} & & \uparrow \text{twisted} \\ A_{q,p}(\widehat{gl_2}) & \xrightarrow{\text{scaling limit}} & A_{\hbar,\eta}(\widehat{gl_2}) \end{array}$$

6 Discussions

In this paper, we propose an elliptic algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ based on a dynamical relations $RLL = LLR^*$, where the dynamical R-matrix is of the $A_1^{(1)}$ type face model. The corresponding (Drinfeld) current algebra is the current algebra generalizing the screening currents for q-Virasoro algebra and is a dynamical twisted algebra of $A_{q,p}(\widehat{gl_2})$, which can be considered as the results of the correspondence between the $A_1^{(1)}$ face model and the eight-vertex model with the generic r . Moreover, the q-primary fields of q-Virasoro Φ_μ and Ψ_μ^* which are also called as the vertex operators for $A_1^{(1)}$ face model, are the intertwining operators of the elliptic algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ at level one.

It is very interesting to extend the present formulation $RLL = LLR^*$ to the case of $A_{n-1}^{(1)}$. The corresponding elliptic algebra is $A_{q,p;\hat{\pi}}(\widehat{gl_n})$. The corresponding (Drinfeld) current algebra of algebra $A_{q,p;\hat{\pi}}(\widehat{gl_n})$ would be the current algebra generalizing the screening currents for q-deformed W_n algebra, which is gen-

erated by $E_j(v), F_j(v)$ and $H_j^\pm(v)$ ($j=1, \dots, n-1$) with the following relations

$$\begin{aligned}
E_i(v_1)E_j(v_2) &= (-1)^{A_{ij}} \frac{[v_1 - v_2 - \frac{A_{ij}}{2}]_r}{[v_1 - v_2 + \frac{A_{ij}}{2}]_r} E_j(v_2)E_i(v_1) \\
F_i(v_1)F_j(v_2) &= (-1)^{A_{ij}} \frac{[v_1 - v_2 + \frac{A_{ij}}{2}]_{r-c}}{[v_1 - v_2 - \frac{A_{ij}}{2}]_{r-c}} F_j(v_2)F_i(v_1) \\
[E_j(v_1), F_j(v_2)] &= \frac{1}{x - x^{-1}} \left\{ \delta(v_1 - v_2 + \frac{c}{2}) H_j^+(v_1 + \frac{c}{4}) - \delta(v_1 - v_2 - \frac{c}{2}) H_j^-(v_1 - \frac{c}{4}) \right\} \\
E_j(v_1)F_{j+1}(v_2) &= -F_{j+1}(v_2)E_j(v_1) \quad , \quad F_j(v_1)E_{j+1}(v_2) = -E_{j+1}(v_2)F_j(v_1) \\
[E_j(v_1), F_l(v_2)] &= 0 \quad , \quad |j - l| > 1 \\
H_i^\pm(v_1)E_j(v_2) &= \frac{[v_1 - v_2 - \frac{A_{ij}}{2} \mp \frac{c}{4}]_r}{[v_1 - v_2 + \frac{A_{ij}}{2} \mp \frac{c}{4}]_r} E_j(v_2)H_j^\pm(v_1) \\
H_i^\pm(v_1)F_j(v_2) &= \frac{[v_1 - v_2 + \frac{A_{ij}}{2} \pm \frac{c}{4}]_{r-c}}{[v_1 - v_2 - \frac{A_{ij}}{2} \pm \frac{c}{4}]_{r-c}} F_j(v_2)H_j^\pm(v_1) \\
H_i^\pm(v_1)H_j^\pm(v_2) &= \frac{[v_1 - v_2 - \frac{A_{ij}}{2}]_r [v_1 - v_2 + \frac{A_{ij}}{2}]_{r-c}}{[v_1 - v_2 + \frac{A_{ij}}{2}]_r [v_1 - v_2 - \frac{A_{ij}}{2}]_{r-c}} H_j^\pm(v_2)H_i^\pm(v_1) \\
H_i^\pm(v_1)H_j^\mp(v_2) &= \frac{[v_1 - v_2 - \frac{A_{ij}}{2} \mp \frac{c}{2}]_r [v_1 - v_2 + \frac{A_{ij}}{2} \pm \frac{c}{2}]_{r-c}}{[v_1 - v_2 + \frac{A_{ij}}{2} \mp \frac{c}{2}]_r [v_1 - v_2 - \frac{A_{ij}}{2} \pm \frac{c}{2}]_{r-c}} H_j^\mp(v_2)H_i^\pm(v_1)
\end{aligned}$$

where $H_j^-(v) = H_j^+(v + \frac{c}{2} - r)$ and the matrix A_{ij} is the Cartan matrix for $A_{n-1}^{(1)}$ Lie algebra

$$A_{ij} = 2\delta_{ij} - \delta_{i+1,j} - \delta_{i-1,j}$$

The above algebraic relations could be derived by the Gauss decomposition of the L^\pm -operators corresponding to the dynamical R-matrix of $A_{n-1}^{(1)}$ face model. We will present the results in the further paper.

Acknowledgements. Some results of this paper was given as a talk by one of authors B.Y.Hou in the XIIth international congress of mathematical physics 13-19 July, 1997, Brisbane, Australia and in the international workshop on “Statistical Mechanics and Integrable Systems” 28 July-8 August 1997, Canberra, Australia. He would like to thank Prof.A.J.Bracken, M.D.Gould, R.J.Baxter and B.M.M.McCoy for their hospitality. He also would like to thank Prof.V.E.Korepin, H.Awata and H.Konno for their interest about this problem in Australia. One of authors W.L.Yang would like to thank Prof.K.J.Shi, L.Zhao and Dr.H.Fan for their helpful discussions.

Note added: After our paper was submitted to the electronic archive, Prof. H.Konno informed us that “ Recently, Jimbo has succeeded to derive the algebra $U_{q,p}(\widehat{sl_2})$ by the Gauss decomposition of a central extended dynamical RLL—relations with the R-matrix introduced by Enriquez and Felder. This

results allows us to clarify a Hopf algebra structure of $U_{q,p}(\widehat{sl_2})$. This work along this line is now in progress”.

Appendix

A. The normal order relation for basic operators

The normal order relations for the screening currents $E(v)$ and $F(v)$ of q-deformed Virasoro algebra are

$$\begin{aligned} E(v_1)E(v_2) &= x^{\frac{4(r-1)v_1}{r}} t(v_2 - v_1) : E(v_1)E(v_2) : \\ F(v_1)F(v_2) &= x^{\frac{4rv_1}{r-1}} t'(v_2 - v_1) : F(v_1)F(v_2) : \\ E(v_1)F(v_2) &= \frac{x^{-4v_1}}{(1 - xx^{2(v_2-v_1)})(1 - x^{-1}x^{2(v_2-v_1)})} : E(v_1)F(v_2) : \\ F(v_2)E(v_1) &= \frac{x^{-4v_2}}{(1 - xx^{2(v_1-v_2)})(1 - x^{-1}x^{2(v_1-v_2)})} : E(v_1)F(v_2) : \end{aligned}$$

The normal order relations for the basic operators in type I and type II vertex operators read as

$$\begin{aligned} \eta_1(v_1)\eta_1(v_2) &= x^{\frac{(r-1)v_1}{r}} g_1(v_2 - v_1) : \eta_1(v_1)\eta_1(v_2) : \\ \eta_1(v_1)\xi(v_2) &= x^{\frac{2(1-r)v_1}{r}} s(v_2 - v_1) : \eta_1(v_1)\xi(v_2) : \\ \xi(v_2)\eta_1(v_1) &= x^{\frac{2(1-r)v_2}{r}} s(v_1 - v_2) : \eta_1(v_1)\xi(v_2) : \\ \xi(v_1)\xi(v_2) &= x^{\frac{4(r-1)v_1}{r}} t(v_2 - v_1) : \xi(v_1)\xi(v_2) : \\ \eta'_1(v_1)\eta'_1(v_2) &= x^{\frac{rv_1}{r-1}} g'_1(v_2 - v_1) : \eta'_1(v_1)\eta'_1(v_2) : \\ \eta'_1(v_1)\xi'(v_2) &= x^{\frac{-2rv_1}{r-1}} s'(v_2 - v_1) : \eta'_1(v_1)\xi'(v_2) : \\ \xi'(v_2)\eta'_1(v_1) &= x^{\frac{-2rv_2}{r-1}} s'(v_1 - v_2) : \eta'_1(v_1)\xi'(v_2) : \\ \xi'(v_1)\xi'(v_2) &= x^{\frac{4rv_1}{r-1}} t'(v_2 - v_1) : \xi'(v_1)\xi'(v_2) : \\ \eta_1(v_1)\xi'(v_2) &= (x^{2v_1} - x^{2v_2}) : \eta_1(v_1)\xi'(v_2) : \\ \xi'(v_2)\eta_1(v_1) &= (x^{2v_2} - x^{2v_1}) : \eta_1(v_1)\xi'(v_2) : \\ \eta'_1(v_1)\xi(v_2) &= (x^{2v_1} - x^{2v_2}) : \eta'_1(v_1)\xi(v_2) : \\ \xi(v_2)\eta'_1(v_1) &= (x^{2v_2} - x^{2v_1}) : \eta'_1(v_1)\xi(v_2) : \\ \xi(v_1)\xi'(v_2) &= \frac{x^{-4v_1}}{(1 - xx^{2(v_2-v_1)})(1 - x^{-1}x^{2(v_2-v_1)})} : \xi(v_1)\xi'(v_2) : \\ \xi'(v_2)\xi(v_1) &= \frac{x^{-4v_2}}{(1 - xx^{2(v_1-v_2)})(1 - x^{-1}x^{2(v_1-v_2)})} : \xi(v_1)\xi'(v_2) : \end{aligned}$$

where

$$\begin{aligned}
g_1(v) &= \frac{\{x^{2+2v}\}\{x^{2+2r+2v}\}}{\{x^{4+2v}\}\{x^{2r+2v}\}} \quad , \quad s(v) = \frac{(x^{2r-1+2v}; x^{2r})}{(x^{1+2v}; x^{2r})} \\
t(v) &= (1 - x^{2v}) \frac{(x^{2+2v}; x^{2r})}{(x^{2r-2+2v}; x^{2r})} \\
g'_1(v) &= \frac{\{x^{2v}\}'\{x^{2+2r+2v}\}'}{\{x^{2r+2v}\}'\{x^{2+2v}\}'} \quad , \quad \{z\}' = (z; x^{2(r-1)}, x^4) \\
s'(v) &= \frac{(x^{2r-1+2v}; x^{2(r-1)})}{(x^{-1+2v}; x^{2(r-1)})} \quad , \quad t'(v) = (1 - x^{2v}) \frac{(x^{-2+2v}; x^{2(r-1)})}{(x^{2r+2v}; x^{2(r-1)})}
\end{aligned}$$

B. The proof of the commutation relations between $K_i^\pm(v)$, $E(v)$ and $F(v)$

The proof is direct substitution the Gauss decomposition of L^\pm -operators in the relations Eq.(24)—Eq.(26)(It should be careful to deal with the order between the dynamical R-matrices and the Gauss components of L^\pm -operators). Here, we give the proof of Eq.(34) as an example. After some straightforward calculation, one can obtain the following commutation relations between the partial currents $E^\pm(v)$ and $F^\pm(v)$

$$\begin{aligned}
[E^\pm(v_2), F^\pm(v_1)] &= K_1^\pm(v_1)^{-1} \frac{c^\pm(v_1 - v_2)}{b^\pm(v_1 - v_2)} K_2^\pm(v_1) - K_2^\pm(v_2) \frac{c'^\pm(v_1 - v_2)}{b'^\pm(v_1 - v_2)} K_1^\pm(v_2)^{-1} \\
[E^-(v_2), F^+(v_1)] &= K_1^+(v_1)^{-1} \frac{c^+(v_1 - v_2 + \frac{c}{2})}{b^+(v_1 - v_2 + \frac{c}{2})} K_2^+(v_1) - K_2^-(v_2) \frac{c'^+(v_1 - v_2 - \frac{c}{2})}{b'^+(v_1 - v_2 - \frac{c}{2})} K_1^-(v_2)^{-1} \\
[E^+(v_2), F^-(v_1)] &= K_1^-(v_1)^{-1} \frac{c^-(v_1 - v_2 - \frac{c}{2})}{b^-(v_1 - v_2 - \frac{c}{2})} K_2^-(v_1) - K_2^+(v_2) \frac{c'^-(v_1 - v_2 + \frac{c}{2})}{b'^-(v_1 - v_2 + \frac{c}{2})} K_1^+(v_2)^{-1}
\end{aligned}$$

Then, we have

$$\begin{aligned}
[E(v_1), F(v_2)] &= K_2^-(v_1 + \frac{c}{2}) \left\{ \frac{c'^+(v_1 - v_2 + \frac{c}{2})}{b'^+(v_1 - v_2 + \frac{c}{2})} - \frac{c'^-(v_1 - v_2 + \frac{c}{2})}{b'^-(v_1 - v_2 + \frac{c}{2})} \right\} K_1^-(v_1 + \frac{c}{2})^{-1} \\
&\quad + K_2^+(v_1) \left\{ \frac{c'^-(v_1 - v_2 - \frac{c}{2})}{b'^-(v_1 - v_2 - \frac{c}{2})} - \frac{c'^+(v_1 - v_2 - \frac{c}{2})}{b'^+(v_1 - v_2 - \frac{c}{2})} \right\} K_1^+(v_1)^{-1}
\end{aligned}$$

Using the following identity

$$\begin{aligned}
\frac{c'^+(v + i\epsilon)}{b'^+(v + i\epsilon)} - \frac{c'^-(v - i\epsilon)}{b'^-(v - i\epsilon)} &= \frac{1}{x - x^{-1}} \delta(v) \frac{[\hat{\pi}]_{r-c}[1]_{r-c}}{\theta'_{r-c}[\hat{\pi} - 1]_{r-c}} \quad \text{When } \epsilon \longrightarrow 0 \\
\theta'_t &= (x - x^{-1}) \frac{\partial}{\partial v} [v]_t|_{v=0}
\end{aligned}$$

we can get the Eq.(34).

C. The proof of the commutation relations Eq.(37)—Eq.(43).

First we prove the relations Eq.(37) and Eq.(38). From the properties Eq.(27) and Eq.(28), using the Gauss decomposition for L^\pm -operators we have

$$\begin{aligned} R^\pm(v_1, \hat{\pi})E(v_2) &= E(v_2)R^\pm(v_1, \hat{\pi} - 2c) \quad , \quad R^\pm(v_1, \hat{\pi})F(v_2) = F(v_2)R^\pm(v_1, \hat{\pi}) \\ R^{*\pm}(v_1, \hat{\pi})E(v_2) &= E(v_2)R^{*\pm}(v_1, \hat{\pi}) \quad , \quad R^{*\pm}(v_1, \hat{\pi})F(v_2) = F(v_2)R^{*\pm}(v_1, \hat{\pi} - 2c) \end{aligned}$$

Notice that the relations Eq.(33a) and Eq.(33b) , we have

$$\begin{aligned} E(v_1)E(v_2) &= \frac{[v_1 - v_2 - 1]_r}{[v_1 - v_2 + 1]_r} E(v_2)E(v_1) \\ F(v_1)F(v_2) &= \frac{[v_1 - v_2 + 1]_{r-c}}{[v_1 - v_2 - 1]_{r-c}} F(v_2)F(v_1) \end{aligned}$$

In order to obtain the relations between $H^\pm(v)$, the relations between $H^\pm(v)$ and $E(v), F(v)$, we should deal with Eq.(30b). It can be rewritten as

$$\frac{a^\pm(v_1 - v_2)}{a'^\pm(v_1 - v_2)} K_2^\pm(v_2) \frac{[v_1 - v_2 + 1]_{r-c} [\hat{\pi}]_{r-c}}{[v_1 - v_2]_{r-c} [\hat{\pi} - 1]_{r-c}} K_1^\pm(v_1)^{-1} = K_1^\pm(v_1)^{-1} \frac{[v_1 - v_2 + 1]_r [\hat{\pi}]_r}{[v_1 - v_2]_r [\hat{\pi} - 1]_r} K_2^\pm(v_2)$$

Taking the limit of $v_1 \longrightarrow v_2$ in both side of the above equation, we have

$$K_2^\pm(v) \frac{[1]_{r-c} [\hat{\pi}]_{r-c}}{\theta'_{r-c} [\hat{\pi} - 1]_{r-c}} K_1^\pm(v)^{-1} = K_1^\pm(v)^{-1} \frac{[1]_r [\hat{\pi}]_r}{\theta'_r [\hat{\pi} - 1]_r} K_2^\pm(v)$$

Then we have two equivalent definition of $H^\pm(v)$

$$\begin{aligned} H^+(v) &= K_2^-(v + \frac{c}{4}) \frac{[\hat{\pi}]_{r-c} [1]_{r-c}}{\theta'_{r-c} [\hat{\pi} - 1]_{r-c}} K_1^-(v + \frac{c}{4})^{-1} \\ &= K_1^-(v + \frac{c}{4})^{-1} \frac{[\hat{\pi}]_r [1]_r}{\theta'_r [\hat{\pi} - 1]_r} K_2^-(v + \frac{c}{4}) \end{aligned} \tag{79}$$

$$\begin{aligned} H^-(v) &= K_2^+(v + \frac{c}{4}) \frac{[\hat{\pi}]_{r-c} [1]_{r-c}}{\theta'_{r-c} [\hat{\pi} - 1]_{r-c}} K_1^+(v + \frac{c}{4})^{-1} \\ &= K_1^+(v + \frac{c}{4})^{-1} \frac{[\hat{\pi}]_r [1]_r}{\theta'_r [\hat{\pi} - 1]_r} K_2^+(v + \frac{c}{4}) \end{aligned} \tag{80}$$

From these two equivalent definitions of $H^\pm(v)$, one can check Eq.(40)—Eq.(41). Here, we give the proof of Eq.(42) as an example

$$\begin{aligned} H^+(v_1)H^+(v_2) &= K_2^-(v_1 + \frac{c}{4}) \frac{[\hat{\pi}]_{r-c} [1]_{r-c}}{\theta'_{r-c} [\hat{\pi} - 1]_{r-c}} K_1^-(v_1 + \frac{c}{4})^{-1} K_1^-(v_2 + \frac{c}{4})^{-1} \frac{[\hat{\pi}]_r [1]_r}{\theta'_r [\hat{\pi} - 1]_r} K_2^-(v_2 + \frac{c}{4}) \\ &= K_2^-(v_1 + \frac{c}{4}) \frac{[\hat{\pi}]_{r-c} [1]_{r-c}}{\theta'_{r-c} [\hat{\pi} - 1]_{r-c}} \frac{a'^-(v_1 - v_2)}{a^-(v_1 - v_2)} K_1^-(v_2 + \frac{c}{4})^{-1} K_1^-(v_1 + \frac{c}{4})^{-1} \frac{[\hat{\pi}]_r [1]_r}{\theta'_r [\hat{\pi} - 1]_r} K_2^-(v_2 + \frac{c}{4}) \end{aligned}$$

$$\begin{aligned}
&= \frac{[v_2 - v_1]_{r-c} [1]_{r-c}}{\theta'_{r-c} [v_2 - v_1 + 1]_{r-c}} a^-(v_2 - v_1) K_2^-(v_1 + \frac{c}{4}) \frac{1}{b'^-(v_2 - v_1)} K_1^-(v_2 + \frac{c}{4})^{-1} \\
&\quad \times K_1^-(v_1 + \frac{c}{4})^{-1} \frac{[\hat{\pi}]_r [1]_r}{\theta'_r [\hat{\pi} - 1]_r} K_2^-(v_2 + \frac{c}{4}) \\
&= \frac{[v_2 - v_1]_{r-c} [v_1 - v_2]_r [1]_{r-c} [1]_r}{\theta'_{r-c} \theta'_r [v_2 - v_1 + 1]_{r-c} [v_1 - v_2 + 1]_r} K_1^-(v_2 + \frac{c}{4})^{-1} \frac{a^-(v_2 - v_1)}{b^-(v_2 - v_1)} K_2^-(v_2 + \frac{c}{4}) \\
&\quad \times K_2^-(v_1 + \frac{c}{4}) \frac{a'^-(v_1 - v_2)}{b'^-(v_1 - v_2)} K_1^-(v_1 + \frac{c}{4})^{-1} \\
&= \frac{[v_1 - v_2 - 1]_r [v_1 - v_2 + 1]_{r-c}}{[v_1 - v_2 + 1]_r [v_1 - v_2 - 1]_{r-c}} H^+(v_2) H^+(v_1)
\end{aligned}$$

here we have used the identity

$$\frac{a'^{\pm}(v_1 - v_2)}{a^{\pm}(v_1 - v_2)} = \frac{a^{\pm}(v_2 - v_1)}{a'^{\pm}(v_2 - v_1)}$$

Similarly, we can prove the other relations among $H^{\pm}(v)$. The following identities are very useful for the proof

$$a^+(v) a^-(-v) = 1 \quad , \quad a'^+(v) a'^-(-v) = 1$$

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